ON A PAPER OF A. FELDZAMEN

BY

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ABSTRACT

Results of A. Feldzamen on semi-similarity of operators are proved here using matrix methods. The use of these methods yields simpler proofs, the formulations of the theorems assume a more transparent form.

The purpose of this note is to give shorter and more transparent proofs of results given in [3]. This will be done by using the methods developed in [4]. It should be mentioned that we assumed separability while Feldzamen does not.

1. **Preliminary notions.** Let S be a normal operator, on a separable Hilbert space H, of uniform multiplicity $n < \infty$. Thus H can be taken as direct sum of n equal spaces $L_2(\Omega, \Sigma, \mu)$, where Ω is a Borel subset of the plane, Σ the collection of Borel subsets of Ω, and μ a finite positive measure.

Also:

$$S\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda f_1(\lambda) \\ \vdots \\ \lambda f_n(\lambda) \end{pmatrix}.$$

See [3], [4], or [5].

The spectral measure $E(.)$ of S is given by

$$E(\delta)\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \begin{pmatrix} \chi(\delta)f_1(\lambda) \\ \vdots \\ \chi(\delta)f_n(\lambda) \end{pmatrix}$$

where $\chi(\delta)$ is the characteristic function of δ.

Every operator A that commutes with S is given by a matrix of bounded and measurable functions $a_{ij}(\lambda)$, where

$$A\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = (a_{ij}(\lambda))\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}.$$

See Theorem 2.1. of [4].

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DEFINITION. The vectors $y_i \in H$, $i = 1 \dots k$, will be called dependent over δ if $y_i(\lambda)$ are dependent for almost every $\lambda \in \delta$.

LEMMA 1.1. *The vectors y_i are dependent over δ if and only if there exist k measurable functions g_i, and a sequence of Borel sets δ_m increasing to δ, such that*

a. The functions g_i are bounded on δ_m and not all zero.

b. If $g_{i,m}$ is the restriction of g_i to δ_m then

$$\sum_{i=1}^{k} g_{i,m}(S)y_i = 0.$$

Proof. It is clear that a. and b. imply dependence. Conversely, let $y_i(\lambda)$ be dependent for $\lambda \in \delta$. For each $\lambda \in \delta$ there exist constants $g_i(\lambda)$ such that

$$\sum_{i=1}^{k} g_i(\lambda)y_i(\lambda) = 0.$$

It is enough to show that one can choose g_i to be measurable. Let us consider the matrix $(y_{i,r}(\lambda))$ where $y_{i,r}(\lambda)$ is the rth component of $y_i(\lambda)$. The set Ω can be decomposed into finitely many disjoint measurable sets, on each a certain determinant of $(y_{i,r}(\lambda))$ is the largest non vanishing one. On each set $g_i(\lambda)$ can be chosen by Cramer's Rule, and are thus measurable.

COROLLARY. *If $k > n$ then the vectors y_i are dependent over every set δ.*

LEMMA 1.2. *Let y_i $1 \leqq i \leqq n$ be independent over Ω. Let x be any vector in H. There exist n measurable functions $f_i(\lambda)$ and a sequence of Borel sets δ_m increasing to Ω such that*

$$x = \lim_{m \to \infty} \sum_{i=1}^{n} f_{i,m}(S)y_i,$$

where $f_{i,m}$ is the restriction of f_i to δ_m and is bounded. The functions f_i are uniquely defined.

Proof. The vectors $x(\lambda)$, $y_i(\lambda)$ are dependent by the previous Corollary. Thus $x(\lambda)$ can be represented by a linear combination of $y_i(\lambda)$. Since these vectors are independent the representation is unique.

The same result could be proved for the case that y_i are independent over some set $\delta \subset \Omega$.

2. **Canonical form for nilpotents.** In this section we will follow [1] to bring a nilpotent matrix with measurable elements to canonical form. It was proved in [4] that if N is quasi nilpotent and commuting with S, then $N(\lambda)^n = 0$ a.e.

Let $A(\lambda; x)$ be an n by n matrix whose elements are polynomials in x with coefficients that are measurable functions of λ. Let Ω_k be the set on which the minimal order of the polynomials $a_{ij}(\lambda; x)$ is equal to k. This is a measurable set.

Let $\Omega_1 = \bigcup_{i,j} \Omega_1^{i,j}$, where $\Omega_1^{i,j} = \{\lambda \mid \text{order of } a_{ij}(\lambda; x) = 1\}$. Again $\Omega_1^{i,j}$ is measurable. An elementary transformation will bring a_{ij} to the upper left corner and by more elementary transformations $A(\lambda; x)$ can be brought to the form

$$\begin{bmatrix} a(\lambda) & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A_1(\lambda; x) & \\ 0 & & & \end{bmatrix}$$

where order of $a(\lambda)$ is one and $A_1(\lambda; x)$ has the same form as $A(\lambda; x)$.

Let us split Ω_k to $\Omega_k^{i,j} = \{\lambda \mid \lambda \in \Omega_k \text{ and order of } a_{ij}(\lambda; x) = k\}$. On $\Omega_k^{i,j}$ we apply to $A(\lambda; x)$ an elementray transformation to bring a_{ij} to the left upper corner. Using the Euclidean Algorithm we see that there are two possibilities:

1. By an elementary transformation (using measurable coefficients) we can bring $A(\lambda; x)$ on $\Omega_k^{i,j}$ to the form

$$\begin{bmatrix} a(\lambda; x) & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A_1(\lambda; x) & \\ 0 & & & \end{bmatrix}$$

where $A_1(\lambda; x)$ has the same form as $A(\lambda; x)$ and $a(\lambda; x)$ divides every element of $A_1(\lambda; x)$.

2. $A(\lambda; x)$ can be transformed to a matrix whose minimal order is less than k, on Ω_k^i .

These considerations prove:

LEMMA 2.1. *There exists a matrix $B(\lambda; x)$ such that both $B(\lambda; x)$ and $B(\lambda; x)^{-1}$ have polynomial elements with coefficients that are measurable functions of λ and*:

$$B(\lambda; x)A(\lambda; x)B(\lambda; x)^{-1} = \operatorname{diag}\{f_1(\lambda; x), f_2(\lambda; x), \dots, f_n(\lambda; x)\},$$

where $f_i(\lambda; x)$ are polynomials in x with measurable coefficients and $f_i(\lambda; x) \mid f_{i+1}(\lambda; x)$.

Let now $A(\lambda; x) = xI - N(\lambda)$ where $N(\lambda)$ represents the nilpotent operator N. Then $f_i(\lambda; x) = x^{i(\lambda)}$ (or 0), for they divide the minimal polynomial of $N(\lambda)$ (Theorem 8, Chapter V, of [1]). Thus $i(\lambda)$ is a measurable function of λ and $0 \leqq i(\lambda) \leqq n$. Let Ω be the union of the disjoint sets Ω_α, where on Ω_α $i(\lambda)$ is equal to a given fixed integer $1 < i \leqq n$. The sets Ω_α are measurable. By chapter V of [1], Theorem 6.10, the matrix $\operatorname{diag}(f_i(\lambda; x))$ is equivalent, on Ω_α, to a canonical Jordan matrix $\operatorname{diag}(f_i(\lambda; x)) \sim xI - Q_\alpha$, where

$$Q_\alpha = \begin{bmatrix} 0\ \varepsilon_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & \varepsilon_{n-1} \\ 0 & \dots & 0 \end{bmatrix}$$

and ε_i is either 1 or zero. Using Lemma 2.1 again one can find a matrix $C(\lambda; x)$, with the same properties as $B(\lambda; x)$ of Lemma 2.1, such that

$$C(\lambda; x)(xI - N(\lambda))\,C(\lambda; x)^{-1} = xI - Q$$

for $\lambda \in \Omega_\alpha$.

Finally by chapter V, Theorem 5.10, of [1]:

$$C_i(Q; \lambda)N(\lambda)C_i(Q_\alpha; \lambda)^{-1} = Q_\alpha$$

when $\lambda \in \Omega_\alpha$.

To summarize:

THEOREM 2.2. *Let N be a nilpotent operator commuting with S and let $N(\lambda)$ be its matrix representation. Let Q_α be the Jordan forms of a nilpotent matrix. There exists a matrix $D(\lambda)$ of measurable functions such that $D^{-1}(\lambda)$ exists, and measurable sets Ω_α whose union is Ω such that*

$$D(\lambda)N(\lambda)D^{-1}(\lambda) = Q_\alpha \qquad \lambda \in \Omega_\alpha.$$

For the matrix Q_α there exist vectors $y_1, \dots, y_r$ such that

$$y_1, Q_\alpha y_1, \dots, Q_\alpha^{j_1-1} y_1, \dots, y_r, Q_\alpha y_r, \dots, Q_\alpha^{j_r-1} y_r$$

are independent, $j_1 + \dots + j_r = n$, and $Q_\alpha^{j_i} y_i = 0$.

Let $x_i(\lambda) = D^{-1}(\lambda)y_i$, and let $\Omega_{\alpha,m} \subset \Omega_\alpha$ be such that $x_i(\lambda)$ is bounded on $\Omega_{\alpha,m}$ and $\Omega_{\alpha,m}$ increases to Ω_α. Then on $\Omega_{\alpha,m}$ (on $E(\Omega_{\alpha,m})H$)

$$N_1, Nx_1, \dots, N^{j_1-1}x_1, \dots, x_r, Nx_r, \dots, N^{j_r-1}x_r$$

are independent, and $N^{j_i}x_i = 0$.

This shows that the sets Ω_α do not depend on the representation of H as direct sum of L_2 spaces (Spectral Multiplicity Theorem).

The sets Ω_α will be called the canonical sets of $S + N$.

3. **Semi similarity.** Let $T = S + N$ and $T_1 = S_1 + N_1$ be two spectral operators (see [2]) and let S have uniform multiplicity n (equivalenty S is similar to a normal operator with uniform multiplicity). In [3] the notion of semi semilarity is defined by:

DEFINITION. T and T_1 are semi similar if there is a sequence of Borel sets δ_m increasing to Ω such that, if $E(.)$ and $E_1(.)$ are the spectral measures of T and T_1, there are bounded maps L_m, from $E_1(\delta_m)H$ to $E(\delta_m)H$, with

$$L_m T_m L_m^{-1} = T_{1m}$$

where $T_m(T_{1m})$ is the restriction of $T(T_1)$ to $E(\delta_m)$ $(E_1(\delta_m))$.

It was shown in [3], Theorem 27, that if T and T_1 are semi similar, then S and S_1 are similar. If T is semi similar to T_1 and $T = KT_2K^{-1}$ for a bounded operator K where T_2 is again spectral then

$$L_m K T_2 K^{-1} L_m^{-1} = T_{1m}$$

or T_2 is semi similar to T_1. Also by the remark following Theorem 2.2 the operators T_2 and T have the same canonical sets.

THEOREM 3.1. *The spectral operators T and T_1 are semi similar if and only if S and S_1 are similar and T and T_1 have the same canonical sets.*

Proof. Without loss of generality we may assume that $S = S_1$. If $S + N$ is semi similar to $S + N_1$ then

$$L_m N_m L_m^{-1} = N_{1m},$$

where N_m and N_{1m} are the restrictions of N and N_1 to $E(\delta_m)H$. But then

$$L_m(\lambda) N_m(\lambda) L_m^{-1}(\lambda) = N_{1m}(\lambda),$$

which proves that $N(\lambda)$ and $N_1(\lambda)$ have the same canonical sets. Conversely, if N and N_1 have the same canonical sets, then on Ω_α

$$N = D^{-1}(\lambda) Q_\alpha D(\lambda), \quad N_1 = D_1^{-1}(\lambda) Q_\alpha D_1(\lambda);$$

hence

$$N_1 = D_1^{-1}(\lambda) D(\lambda) N(\lambda) D^{-1}(\lambda) D_1(\lambda).$$

Define δ_m so that $D^{-1}(\lambda)D_1(\lambda)$ and $D_1^{-1}(\lambda)D(\lambda)$ be bounded on δ_m, and $L_m(\lambda) = \{D_1^{-1}(\lambda)D(\lambda) \mid \text{restricted to } E(\delta_m)H\}$.

COROLLARY. *Semi similarity is a transitive relation.*

This is Theorem 26 of [3]. Theorem 3.1 is essentially equivalent to Theorem 29, and 30 of [3].

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